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Integral and Rational Completions of Combinatorial Matrices

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Earlier results by Marshall Hall on integral completions of matrices satisfying orthogonality conditions are extended as far as possible, with special attention given to the Hadamard case. A result on restricting the denominators of rational completions to a power of 2 is also given.

INTRODUCTION

Marshall Hall [1] was able to show that, given certain existence conditions, any integral $(n - r) \times n$ matrix X satisfying $XX^T = mI$ can be extended rationally to an $n \times n$ matrix Z, with X as its first $n - r$ rows and satisfying $ZZ^T = mI$. Furthermore, in the case where $r \leqslant 2$, Z may in fact be taken to be integral.

In the case of Hadamard matrices, Hall was further able to show that in the case of $r \leqslant 4$, any $(1, -1)$ matrix X satisfying $XX^T = nI$ can in fact be extended to a Hadamard matrix Z, with X as its first $n - r$ rows, provided only that the existence condition of $n \equiv 0 \pmod 4$ is satisfied.

The purpose of this paper is to extend both the limits above to $r \leqslant 7$ and, further, to give a condition for the denominators of the nonintegral elements of Z for arbitrary r.

1. EXISTENCE CONDITIONS FOR RATIONAL COMPLETIONS

For completeness, two theorems from [1] are quoted:

THEOREM 1.1. *There exists an integral square matrix A of order n such that $AA^T = mI$, m a positive integer if and only if*

(i) *for n odd, m is a square;*
(ii) *for $n \equiv 2 \pmod 4$, m is the sum of two integral squares;*
(iii) *for $n \equiv 0 \pmod 4$, m is any positive integer.*

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THEOREM 1.2. *Suppose that X is an $(n-r) \times n$ integral matrix satisfying $XX^T = mI_{n-r}$, and that m and n satisfy the conditions of Theorem* 1.1. *Then there is an $n \times n$ matrix Z having X as its first $n-r$ rows and having its last r rows rational such that $ZZ^T = mI_n$.*

Proofs may be found in [1], which generally follow from the results of [2].

2. INTEGRAL COMPLETIONS

THEOREM 2.1. *Let m, n satisfy the conditions of Theorem* 1.1. *If $r \leqslant 7$, X is an integral $(n-r) \times n$ matrix satisfying $XX^T = mI$, then there is an $n \times n$ integral matrix A with X as its first $n-r$ rows such that $AA^T = mI$.*

Proof. By Theorem 1.2, there is a rational completion of X, call it

$$A_0 = \begin{bmatrix} X \\ Y^T \end{bmatrix}, \tag{2.1}$$

where Y is a rational $n \times r$ matrix. Let s be the smallest integer so that sA_0 is integral, giving

$$A_1 = \begin{bmatrix} sX \\ sY^T = Y_1^T \end{bmatrix}. \tag{2.2}$$

Let p be a prime dividing s. Then $Y_1Y_1^T \equiv 0 \pmod{p^2}$, so that Y_1 has rank at most $\frac{1}{2}r$ over Z_p. Suppose that $v_1, v_2, ..., v_k$ are row vectors from Y_1 which form a basis of the row space of Y_1 over Z_p. The dual of $\langle v_1, ..., v_k \rangle$ is of rank $r-k$, but $v_1, ..., v_k$ are self-dual over Z_p. Hence there are $r-2k$ further vectors $v_{k+1}, ..., v_{r-k}$ which together with $v_1, ..., v_k$ form a basis of $\langle v_1, ..., v_k \rangle^{\perp}$. Finally, there are k further vectors $v_{r-k+1}, ..., v_r$ which complete a basis of $Z_p{}^r$.

If we now consider these vectors as elements of Q^r, it is clear that if they are independent over Z_p, they are independent over Q. Form the following r by r matrix B:

$$B = \begin{bmatrix} v_1 \\ \cdot \\ v_k \\ pv_{k+1} \\ \cdot \\ pv_{r-k} \\ p^2v_{r-k+1} \\ \cdot \\ p^2v_r \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_r \end{bmatrix}. \tag{2.3}$$

B is evidently nonsingular. We have $p^{r-2k}p^{2k} = p^r$ dividing $\det B$. However, upon removing the factors p and p^2 above, the independence of the rows over Z_p ensures that $p^{r+1} \nmid \det B$.

We have $p^2 \mid b_i \cdot b_j$, all $1 \leqslant i \leqslant j \leqslant r$. This clearly holds if $j > r-k$ as $p^2 \mid b_j$. If $k < j \leqslant r-k$, then if also $k < i \leqslant r-k$, $p \mid b_i$, $p \mid b_j$ implies $p^2 \mid b_i \cdot b_j$. If $1 \leqslant i \leqslant k$, then $v_j \in \langle v_1, ..., v_k \rangle^{\perp}$ implies $p \mid v_i \cdot v_j = b_i \cdot v_j$ and so $p^2 \mid b_i \cdot b_j = v_i \cdot v_j$ as v_i, v_j are rows of Y_1. Hence $BB^T \equiv 0 \pmod{p^2}$.

Let y be a row of Y_1. Since $v_1, ..., v_k$ is a basis of the row space of Y_1 over Z_p, we may write $y = \sum_{i=1}^{k} a_i v_i + py'$, where the a_i are integers and y' is another integral vector. We have $p^2 \mid v_j \cdot y$, $j = 1, ..., k$, as y is a row vector from Y_1. Thus

$$p^2 \,\Big|\, \sum_{i=1}^{k} a_i (v_j \cdot v_i) + p(v_j \cdot y'), \qquad j = 1, ..., k. \tag{2.4}$$

But $p^2 \mid v_j \cdot v_i$, $i, j = 1, ..., k$. Thus

$$p \mid v_j \cdot y', \qquad j = 1, ..., k. \tag{2.5}$$

Evidently, $y' \in \langle v_1, ..., v_k \rangle^{\perp} = \langle v_1, ..., v_{r-k} \rangle$ over Z_p. This gives $y = \sum_{i=1}^{k} c_i v_i + p(\sum_{i=k+1}^{r-k} c_i v_i) + p^2 y''$. Thus we have

$$y \equiv \sum_{i=1}^{r-k} c_i b_i \pmod{p^2}. \tag{2.6}$$

It follows from (2.6) that

$$b_j \cdot y \equiv 0 \pmod{p^2}, \qquad j = 1, ..., r, \tag{2.7}$$

and so

$$BY_1^T \equiv 0 \pmod{p^2}. \tag{2.8}$$

Let $q \neq p$ be a prime dividing $\det B$. There are integers $d_1, ..., d_r$, with g.c.d. $(d_1, ..., d_r) = 1$ so that $q \mid \sum_{i=1}^{r} d_i b_i$. It is well known that with this condition, there is an integral matrix D of determinant 1 with $(d_1, ..., d_r)$ as its first row.

Let $B' = DB$. Then $B' Y_1^T = DBY_1^T \equiv 0 \pmod{p^2}$. Likewise, $B'(B')^T = DBB^T D^T \equiv 0 \pmod{p^2}$. We may write

$$B' = \begin{bmatrix} qb_1' \\ b_2' \\ \cdot \\ b_r' \end{bmatrix}, \tag{2.9}$$

where all the b_i' are integral. $p^2 \mid (qb_1') \cdot (qb_1') = q^2(b_1' \cdot b_1')$ implies $p^2 \mid b_1' \cdot b_1'$. $p^2 \mid (qb_1') \cdot b_i'$ implies $p^2 \mid b_1' \cdot b_i'$. Similarly, if y is a row of Y_1, $p^2 \mid (qb_1') \cdot y$ implies $p^2 \mid b_1' \cdot y$. Divide out the factor of q from the first row of B' and call the resulting matrix B''. The preceding argument shows that

$B''(B'')^T \equiv 0 \pmod{p^2}$ and $B''Y_1^T \equiv 0 \pmod{p^2}$. Furthermore, $\det B'' = q^{-1} \det B' = q^{-1} \det B$ as $\det D = 1$. This means that $p^r \mid \det B''$ but $p^{r+1} \nmid \det B''$ as before.

Continue inductively until we arrive at a matrix C, with $CC^T \equiv 0 \pmod{p^2}$, $CY_1^T \equiv 0 \pmod{p^2}$, and $\det C = p^r$.

Consider the integral matrix $E = p^{-2}CC^T$. This will be an integral, positive definite quadratic form of determinant $(p^{-2})^r(p^r)^2 = 1$. Hence, as $r \leqslant 7$ by hypothesis, E is integrally equivalent to the identity [3]. There is an integral matrix D_1 of determinant 1 satisfying $D_1ED_1^T = I_r$. Hence, if we let $C_1 = D_1C$, $C_1C_1^T = p^2(D_1ED_1^T) = p^2I_r$. Furthermore, we still have $C_1Y_1^T = D_1CY_1^T \equiv 0 \pmod{p^2}$.

Let $C_2 = pI_{n-r} \oplus C_1$. Consider

$$A_2 = C_2A_1 = \begin{bmatrix} psX \\ C_1Y_1^T \end{bmatrix}. \tag{2.10}$$

We have $A_2A_2^T = C_2(A_1A_1^T)\,C_2^T = C_2(s^2mI_n)\,C_2^T = p^2s^2mI_n$. Write $s_1 = ps_1$, $C_1Y_1^T = p^2Y_2^T$ (Y_2 integral as $C_1Y_1^T \equiv 0 \pmod{p^2}$). Then

$$A_3 = p^{-2}A_2 = \begin{bmatrix} s_1X \\ Y_2^T \end{bmatrix} \tag{2.11}$$

and we have $A_3A_3^T = p^{-4}(p^2s^2mI_n) = s_1^2mI_n$ with $s_1 < s$.

With induction on s satisfied, we conclude that s is minimal only for $s = 1$, in which case the integral matrix A thereby determined satisfies $AA^T = mI_n$ and has X as its first $n - r$ rows. That $r \leqslant 7$ is best possible may be seen by taking $n = 9$, $m = 9$, $r = 8$, and letting

$$X = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]. \tag{2.12}$$

We have $(3I_9)(3I_9)^T = 9I_9$, and yet there are no integral row vectors of length 9 which are both orthogonal to X and have sum squares of the entries equal to 9. This follows easily from the fact that there are no such solutions mod 2.

There is, however, a half-integral solution:

$$A = \frac{1}{2}\begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & -3 & -3 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3 & 3 & -3 & 0 & 0 & 0 & 0 & 0 \\ 3 & -3 & -3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 3 & -3 & -3 & 3 & 0 \\ 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -4 \\ 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & 4 \end{bmatrix}. \tag{2.13}$$

The question arises as to what general restrictions can be placed of the rational elements of a completion, if in fact no integral completion is possible, as above.

We have

THEOREM 2.2. *Let m, n satisfy the conditions of Theorem* 1.1. *If X is an integral $(n-r)\times n$ matrix satisfying $XX^T = mI$, there is an $n\times n$ rational matrix A with X as its first $n-r$ rows and the remaining elements having denominators with no nontrivial odd factors* (*i.e.*, *2^eA is integral for sufficiently large e*).

Proof. We may find a rational completion A_0 of X as in (2.1). Let 2^as (s odd) be the smallest integer so that 2^asA is integral, giving

$$A_1 = \begin{bmatrix} 2^asX \\ 2^asY^T = Y_1^T \end{bmatrix}. \tag{2.14}$$

Let q be an odd prime dividing s. Then, as outlined in the proof of Theorem 2.1, we may determine an integral r by r matrix C with $CC^T \equiv 0 \pmod{q^2}$, $CY_1^T \equiv 0 \pmod{q^2}$, $\det C = q^r$. The associated integral quadratic form $E = q^{-2}CC^T$ is again positive definite of determinant 1. E is 2-rationally congruent to the identity (qC^{-1} consists of elements with odd denominators and $(qC^{-1})E\,(qC^{-1})^T = I$). Hence, it can be shown that E is p-adically equivalent to the identity for arbitrary prime p [4]. If $r \geqslant 4$, E p-adically represents all positive integers for arbitrary p. In particular, E p-adically represents 4^b for arbitrary b, p. Assume $r \geqslant 5$. Then a theorem of Tartakowski [5] shows that there is a constant G_E such that if $N > G_E$ and a positive form E p-adically represents N for all primes p, then E integrally represents N. Choose b so large that $4^b > G_E$. Then here E integrally represents 4^b. Let w be a $1\times r$ integral row vector satisfying $wEw^T = 4^b$. Then $(wC)(wC)^T = q^24^b$. Let $w_1 = wC$. $w_1Y_1^T = wCY_1^T \equiv 0 \pmod{q^2}$. Let $w_2 = q^{-2}w_1Y_1^T$. Then $w_2\cdot w_2 = q^{-4}w_1Y_1^TY_1w_1^T = q^{-4}w_1(4^as^2mI_r)\,w_1^T = q^{-4}4^as^2m(q^24^b) = q^{-2}(4^{a+b}s^2m)$. Further, $Xw_2^T = q^{-2}XY_1w_1^T = 0$. Write

$$X_1 = \begin{bmatrix} 2^{a+b}s_1X \\ w_2 \end{bmatrix}, \tag{2.15}$$

where $s_1 = q^{-1}s$. Then $X_1X_1^T = 4^{a+b}s_1^2mI_{n-r+1}$.

If $r < 5$, X has an integral completion. Let w be one of the integral rows completing X and let $w_2 = 2^as_1w$. Then if we set $b = 0$, X_1 formed as in (2.15) has the same properties as above.

Assume inductively that the result holds for an $r - 1$ row completion, arbitrary s, a. Then we may form the matirx

$$A_2 = \begin{bmatrix} 2^{a+b}s_1X \\ W_2 \\ Y_2^T \end{bmatrix}, \tag{2.16}$$

where the denominators of the elements of Y_2 are divisors of 2^c, some c. Then the matrix

$$A_3 = \begin{bmatrix} 2^{a+b+c}s_1X \\ 2^cW_2 \\ 2^cY_2^T \end{bmatrix} \tag{2.17}$$

is integral, with $A_3A_3^T = 4^{a+b+c}s_1^2mI$ and $s_1 < s$. By induction, there is a solution with $s = 1$,

$$A_4 = \begin{bmatrix} 2^eX \\ Y_3^T \end{bmatrix}, \tag{2.18}$$

for e sufficiently large, with A_4 integral. $A = 2^{-e}A_4$ gives the desired rational completion.

It will be noted that the prime $p = 2$ is unique with respect to the property of affording rational completions of an arbitrary X with denominators powers of p. From our previous example of X in (2.12), parity conditions will still exclude any integral orthogonal vectors, irrespective of multiplication of X by any odd factor, in particular p^e for some odd prime p.

3. Hadamard Completions

The problem of completing appropriate $(1, -1)$ matrices to Hadamard matrices is probably the most interesting subcase of the preceding, combinatorially. It will be noted, however (see [1]), that the existence of even an integral completion does not imply the existence of a $(1, -1)$ completion. Further, counterexamples exist (see [1]) to show that the completion of the last seven rows is the most that could possibly be expected in general. Hall was able to deal with the case of $r \leqslant 4$, viz.:

Theorem 3.1. *Let X be an $n - r \times n$ matrix, $n \equiv 0 \pmod 4$ with every entry $+1$ or -1 and satisfying $XX^T = nI_{n-r}$. Then if $r \leqslant 4$, there is a Hadamard matrix of order n with X as its first $n - r$ rows.*

To deal with the cases $r = 5$, 6, or 7, we need one additional theorem and three simple lemmas.

We quote from [1].

THEOREM 3.2. *Suppose that $AA^T = D_1 \oplus D_2$, where A is of order n and nonsingular, and D_1 and D_2 are of order r and $n - r$ and are nonsingular. Suppose further that X and Y are $r \times n$ matrices such that $XX^T = YY^T = D_1$. Then there exists an orthogonal matrix U of order n such that $XU = Y$. This result holds for all fields F of characteristic different from* 2.

Proof. This follows from [2, Theorem 2.1] and is proved in [1].

The lemmas needed are the following.

LEMMA 3.3. *Let X satisfy the conditions of Theorem* 3.1 *with r unspecified, and suppose Y is a rational completion of X; i.e.,*

$$A_0 = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{3.1}$$

satisfies $A_0 A_0^T = nI_n$. Then the entries of Y^TY are integers of the same parity as r of absolute value no larger than r.

Proof. Since $A_0^T A_0$ is integral and X^TX is integral, then $Y^TY = A_0^TA_0 - X^TX$ is also integral. Furthermore, n is even by assumption, so $A_0^TA_0 \equiv 0 \pmod 2$, and since all the entries of X are $+1$ or -1, $X^TX \equiv (n-r)J \pmod 2$. So $Y^TY \equiv rJ \pmod 2$. Since the main diagonal entries of $A_0^TA_0$ are n and the main diagonal entries of X^TX are $n - r$, the main diagonal entries of Y^TY are r. Applying the Schwarz inequality to the rows of Y^T shows that the absolute value of any inner product is no larger than r.

LEMMA 3.4. *Call $B = (b_{ij}) = Y^TY$, where Y is as in Lemma* 3.3. *Then for arbitrary i, j, k, $b_{ij} + b_{ik} + b_{jk} \equiv -r \pmod 4$.*

Proof. Let x^T, y^T, z^T be three columns from X. Then $x \cdot x = y \cdot y = z \cdot z = n - r$. Let $w = x + y + z$. Then, since x, y, z have all odd entries, so does w. Thus $w \cdot w \equiv n - r \pmod 8$ as any odd square is congruent to $1 \pmod 8$. Then $(x \cdot y) + (x \cdot z) + (y \cdot z) = \frac{1}{2}(w \cdot w - x \cdot x - y \cdot y - z \cdot z) \equiv -(n - r) \equiv r \pmod 4$, as $n \equiv 0 \pmod 4$. Thus, if $C = (c_{ij}) = X^TX$, we have $c_{ij} + c_{ik} + c_{jk} \equiv r \pmod 4$ for any i, j, k. Hence $B \equiv -C \pmod 4$ satisfies the condition stated.

LEMMA 3.5. *Suppose that one of the off-diagonal entries of Y^TY is $r - 2$ or $-(r - 2)$. Then there is an orthogonal transformation U so that UY has a row composed entirely of entries $+1$ or -1.*

Proof. We may suppose that the first and second columns of Y have the specified inner product. If the inner product is negative, negate one of the columns, so that we may suppose that the inner product is, in fact, $r - 2$.

The same orthogonal transformation will suffice. Call the two columns x^T, y^T. Then we have

$$W = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad WW^T = \begin{bmatrix} r & r-2 \\ r-2 & r \end{bmatrix} \tag{3.2}$$

so that WW^T is nonsingular. Let $D_1 = WW^T$. Adding further rows to W, orthogonal to x and y, it is easy to see that we may obtain a nonsingular rational $r \times r$ matrix W' with $W'(W')^T = D_1 \oplus D_2$, some D_2. Now let

$$Z = \begin{bmatrix} 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \tag{3.3}$$

where Z is $2 \times r$. Then $ZZ^T = D_1 = WW^T$. All the hypotheses of Theorem 3.2 are now satisfied and there is a rational orthogonal U so that $WU = Z$ and hence, $U^T W^T = Z^T$. Let $Y_1 = U^T Y$. Then the first two columns of Y_1 are the rows of Z and $Y_1 Y_1{}^T = U^T Y Y^T U = nI_r$ and also $Y_1 X^T = U^T Y X^T = 0_r$. Hence, if we take

$$A_1 = \begin{bmatrix} X \\ Y_1 \end{bmatrix}, \tag{3.4}$$

then $A_1 A_1{}^T = nI_n$ as before. Now let z^T be any other column of Y_1 and let s be its last entry. Evidently, $a \cdot z - b \cdot z = 2s$. Lemma 3.4 evidently still holds if we replace b by its negative, and we obtain

$$a \cdot z - b \cdot z - a \cdot b = 2s - (r - 2) \equiv -r \pmod 4. \tag{3.5}$$

So s is integral and $2s \equiv 2 \pmod 4$ or s is odd. Evidently the last row of Y_1 consists of n odd integers, the sum of whose squares is n. This forces all entries of the last row to be $+1$ or -1 as desired.

In what follows, we assume that the columns of A_0 are permuted so that the first r columns of Y are independent. We will not hesitate to negate columns of A_0 as seems most appropriate, as this operation can be reversed if an integral Hadamard completion is found. Further, if we denote the first q (independent) columns of Y by W^T, then if we have any other rational matrix Z satisfying $ZZ^T = WW^T = D_1$, which is nonsingular, then by the procedure outlined in the proof of Lemma 3.5, Y is orthogonally transformable into a new rational matrix Y', with Z^T as the first q columns and which gives another rational completion of X.

Case 1: $r = 5$. Let A_0, Y be as in (3.1), the existence of the rational completion being assured by Theorem 1.2 .Let $B = (b_{ij}) = Y^T Y$. Then the entries of B must be, by Lemma 3.3, 1, -1, 3, -3, 5, or -5. If any entry is 3 or -3, then by Lemma 3.5, we may apply an orthogonal transformation to obtain a row of $+1$, -1 entries, and this then reduces to the case $r = 4$, settled in Theorem 3.1. So suppose there are no entries 3, -3 in B. Let W^T be the 5×5 matrix consisting of the first five columns of Y (assumed to be independent). Then $C = WW^T$ has all off-diagonal entries $+1$ or -1. Negate appropriate rows of W so that the first row and column of C (except for $c_{11} = 5$) consists of all $+1$ entries. Since $-r \equiv 3 \pmod 4$, Lemma 3.4 now implies that the remaining off-diagonal elements of C are all $+1$, i.e., $C = 4I_5 + J_5$. Let

$$Z = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 \end{bmatrix}. \tag{3.6}$$

Then $ZZ^T = C$. Apply an orthogonal transformation to Y to obtain Y' with first five columns Z^T. Let y be another column of Y'. If Zy has an entry 5 or -5, it must be identical to or a negative of a column of Z^T. Otherwise all entries are $+1$ or -1. Lemma 3.4 now implies that all entries are identical, so negating if necessary, we may take them all to be $+1$. We may solve directly for y and find that

$$y^T = \tfrac{1}{3}(1 \quad 1 \quad 1 \quad 1 \quad -1). \tag{3.7}$$

Evidently, $y^T y = \frac{5}{9} \neq 5$ and so this case cannot arise. Thus Y' is entirely $(1, -1)$ and a Hadamard completion has been found.

Case 2: $r = 6$. Let A_0, Y be as in (3.1). The entries in $B = Y^T Y$ must come from 0, 2, -2, 4, -4, 6, -6. If any entry is 4 or -4, then using Lemma 3.5, we may reduce to the previous case. Let W^T be the 6×6 matrix consisting of the first six (independent) columns of Y. Then if $C = WW^T$, all off-diagonal entries are 0, 2, or -2. At this point a certain amount of hand and computer calculation is involved. Up to permutation and negation of columns of W^T, we find 28 possible C satisfying Lemma 3.4. Of these, 19 fail to have as determinants perfect integral squares (necessary if there is a rational solution W as above). For the remaining 9, $(1, -1)$ Z satisfying $ZZ^T = C$ are found in all cases. Let y be another column of Y and let $v = Zy$. v must be consistent with Lemma 3.4. In addition, $y^T y = 6$ implies $(Z^{-1}v)^T(Z^{-1}v) = v^T C^{-1} v = 6$. Taking these two conditions into account, we find that for all admissable, v, $y = Z^{-1}v$ is $(1, -1)$. So Hadamard completions exist in all cases. Tables of C, Z, v, y are found in [6].

Case 3: $r = 7$. Let A_0, Y be as in (3.1). The entries in $B = Y^T Y$ must come from 1, -1, 3, -3, 5, -5, 7, -7. If any entry is 5 or -5, then using Lemma 3.5, we may reduce to the previous case. Let W^T be the 7×7 matrix consisting of the first seven (independent) columns of Y. Then if $C = WW^T$, all off-diagonal entries are 1, -1, 3, or -3. A computer is used to find all possibilities for C, up to permutation and negation of the rows of W, satisfying Lemma 3.4 and with determinant a perfect integral square. A total of 167 possibilities is found. In all cases, one or more $(1, -1)$ matrices Z are found so that $ZZ^T = C$.

Let y be a further column of Y and suppose that $v = Zy$. Then as in the previous case, we must have $v^T C^{-1} v = r = 7$. If any entry in v is 7 or -7, then Y is identical to or a negative of a column of W^T. Let then $y_1, \ldots, y_k$ be the remaining columns of Y, with no entry of $v_i = Zy_i$ equal to 7 or -7, all i. Negating if necessary, we may assume that the first entry of v_i is 1 or -3 in all cases. Lemma 3.4 shows that if z is the first column of Z^T, then $(z \cdot y_i) + (z \cdot y_j) + (y_i \cdot y_j) \equiv 1 \pmod 4$. But we now have $z \cdot y_i \equiv z \cdot y_j \equiv 1 \pmod 4$, all i, j, Hence we now have $y_i \cdot y_j \equiv -1 \pmod 4$, all i, j.

Let C be fixed. For each $(1, -1)$ Z satisfying $ZZ^T = C$, determine the subset $V_Z = \{v: v$ compatible with Lemma 3.4 and $Z^{-1}v$ is $(1, -1)\}$. Let $V_1, \ldots, V_a$ be all such distinct subsets. Suppose then an incompatible subset $V = \{v_1', \ldots, v_k'\}$ of v_i' individually compatible with Lemma 3.4 is found, further satisfying $(v_i')^T C^{-1} v_j'$ $(= y_i' \cdot y_j') \equiv -1 \pmod 4$, all i, j. Then clearly such a subset exists with $k \leqslant a$. Since a is typically rather small, it was possible to use the computer to check through all such subsets, for all C. It was found that no such incompatible V existed for any C.

Hence a $(1, -1)$ completion is always possible in the $r = 7$ case as well. Tables of C, Z, v, y are also to be found in [6].

Thus we can now summarize these results in the following theorem.

THEOREM 3.6. *Let X be an $n - r \times n$ matrix, $n \equiv 0 \pmod 4$ with every entry $+1$ or -1 satisfying $XX^T = nI_{n-r}$. Then if $r \leqslant 7$, there is a Hadamard matrix of order n with X as its first $n - r$ rows.*

It seems likely that there should be a completely theoretical proof of this result; however, this has yet to be found.

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